JUGGLOLOGY!

Steven Spallone
Associate Professor of Mathematics
Indian Institute of Science Education and Research, Pune.

An **Exciting Science** Talk
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What is Jugglology?

Jugglology: (noun) the scientific study of juggling, usually involving the mathematical modeling of complex patterns.

When he understood the jugglology properly, he discovered new tricks.

Jugglology was invented in the early '80s by jugglers trying to describe their tricks on internet newsgroups, before the days of YouTube.

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By the laws of physics, property 4 implies that the tosses spend the same amount of time in the air.

Juggling Numbers

By Properties 1. and 2. we have a sequence of alternating tosses.

L R L R L R L R...

Each toss will land some number of beats later. Write down that number at each toss. For example, the 3-ball cascade is:

3 3 3 3 3 3 3 3...

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The 3-ball shower is:

5 2 5 1 5 1 5 1 5 1 5...

Decryption

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- **5: big toss across**
- **6: big toss up**

Et cetera...

Beginner's Corner

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- **1111111…**
- **404040...**
- **242424…**
- **222222…**
- **3131313…**
- **2313131…**
- **212121...**

Oops!

Not every sequence is jugglable! A randomly chosen sequence will probably have a collision.

Can you find a sequence which forces *3* balls collide at once?

Guess the Pattern

I'll juggle some patterns, and you try to guess what the numbers are.

Let's also give the juggling diagrams!

Puzzle on the horizon: Find a formula so that if you input a repeating juggling sequence, it outputs how many balls it takes.

Pro Tips: Special Features of 0 and 1

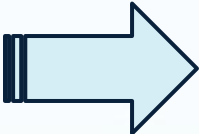- If your pattern has a 0, you can snap your fingers.

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501 ➡ 531

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- Or...you can add a ball!

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- If your pattern has a 1, you can handoff the 1 behind the back or under your leg.

441, 501, 413,...

Back to the 4 Ball Cascade Problem

Remember the Cascade Rules:
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One ball cascade: 11111...

Three ball cascade: 33333...

Five ball cascade: 55555…   etc.

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So...4 ball cascade impossible. **SO STOP ASKING!**

Other Famous Families

Fountains:
2-ball: 222222…
4-ball: 444444…
6-ball: 666666…

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4-ball: ?

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4-ball: 717171…

Half-Shower Rule:

Must be of the form:

...abababab...

Which are jugglable? How many balls do they use?

How many balls?

Find the pattern:

- 333... : three balls
- 5151... : three balls
- 5353... : four balls
- 4040... : two balls
- 424242... : three balls
- 501501... : two balls

Average Theorem

For a periodic jugglable sequence, the number of balls is the average of the numbers.

Juggtology Corollary: *If a (periodic) sequence is jugglable, then the average must be an integer!*

Is the converse true? If the average is an integer, must the sequence be jugglable?

No...but Yes

321321… has average 2, but is not jugglable. *But* you can "juggle" the numbers to get a jugglable pattern:

312312...

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The average of 1, 2, 2, 5, 5 is 3. Can you juggle the numbers to get a jugglable pattern?

New Patterns from Old

You can add the period to any of the numbers and get a new jugglable sequence!

201201201… has period 3.

Add 3 to 0 to get:

231231231…

Add 3 to 2 to get:

501501501…

Note: Every time you add 3 the pattern requires another ball!

How Far Can You Push It?

Since 201201... is jugglable, so is any abcabc... as long as:

- 2 is the remainder when you divide a by 3.
- b is a multiple of 3.
- 1 is the remainder when you divide c by 3.

"(a,b,c) is congruent to (2,0,1) modulo 3."

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3 Ball Examples:
Idle: 234234...
Trampoline: 504504…

Can you find more 3 Ball Examples?

Breaking the Rules: Multiplex

Tossing/Catching two or more balls at once: Use brackets.

Cascade with stacking: [33]33[33]33...
Cascade with splits: [ab]c[ab]c...

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Cascade with two splits: [32][32]...

Diagram these!

Hmmm...Is there an *AVERAGE THEOREM* for multiplex?

The numbers 1, 2, 2, 5, 5 can be juggled to give:

52512

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THANK YOU!

- To learn more about jugglology, read "The Mathematics of Juggling" by Burkard Polster.
- Or search the internet for "siteswap notation".