Polynomials that no one can solve!

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Type	Example	Degree
Linear	$X + 1$	1
Quadratic	$X^2 + 2X + 1$	2
Cubic	$X^3 + 2X$	3
Quartic	$2X^4 + x^3 + 2$	4
Quintic	$X^5 + 1$	5

History of polynomials

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Problem: Given a specific area, they were unable to calculate lengths of the sides of certain shapes, and without these lengths they were unable to design floor plan for their kingdom.

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  $p(5) = (5 - 5)^2 = 0$,

  So, 5 is a root of $(X - 5)^2$.

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$$(4)^2 = 16 \text{ so, we write } \sqrt{16} = 4.$$

The sign $\sqrt{\cdot}$ is called a **radical sign**.

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In general, for a quadratic equation $aX^2 + bX + c$, there are two roots;

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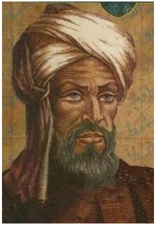

$$x = \frac{-b + (\sqrt{b^2 - 4ac})}{2a}, \qquad x = \frac{-b - (\sqrt{b^2 - 4ac})}{2a}$$

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Gerolamo Cardano (Italy, 600 years ago) first published the formula for roots of cubic polynomials

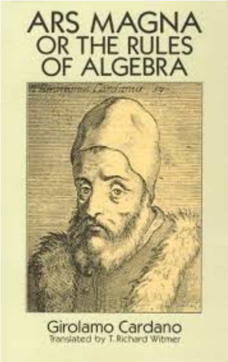

Figure: Gerolamo Cardano (1501-1576)

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Consider a cubic equation

$$aX^3 + bX^2 + cX + d = 0$$

Formula for a root of this polynomial is given by

$$\begin{aligned} x \;=\; & \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \\ & + \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \;-\; \frac{b}{3a}. \end{aligned}$$

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What happens to quartic polynomials?

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There is a analogous formula for finding roots given by **Lodovico Ferrari** (1545), a student of Cardano but it is much worse and I won't even try to write it down here!!

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Allegedly he was in love with a woman engaged to an artillery officer. Galois challenged the officer in order to win her affection. In the nights leading up to the duel, he did write many letters to his friends/colleagues. The most significant of these had the ideas which are foundation of Galois Theory which brought to light non-solvability of a general quintic.

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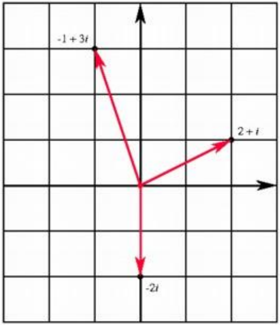

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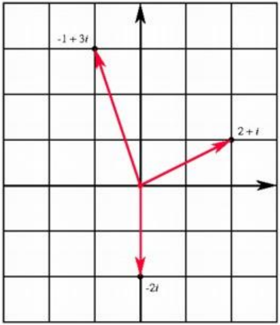

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'Every polynomial $X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ of degree $n > 0$ where $a_i's$ are complex numbers, has a root in complex numbers'.

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Also, there is an integer $-m$ such that $m + (-m) = 0$.

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Example: the set of integer numbers

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Example: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

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He proved that roots of a general quintic polynomial can not be expressed in terms of radicals whenever the associated group is 'non-solvable'.

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One such polynomial is $X^5 - 4X + 2$.

Timeline

3700 years ago	Egyptians made a table.
3600 years ago	Pythagoras worked with integers and rational numbers.
2500 years ago	Babylonians solved some quadratic equations.
1100 years ago	Al-Khwarizmi proved a formula for roots of a quadratic.
600 years ago	Cardano gave a formula for finding roots of a cubic.
600 year ago	Ferrari proved a formula for roots of quartic polynomials.
217 years ago	Gauss proved Fundamental theorem of Algebra.
200 years ago	Galois proved non-solvability of some quintic polynomials.

Thank You!